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A LINEAR PROGRAMMING MODEL FOR LAND-USE ALLOCATION

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I. INTRODUCTION

The allocation of a parcel of land to a mixture of nonexclusive uses by real estate decision-makers has traditionally relied on subjective evaluation and intuitive judgment. Most proposed objective models depend upon optimizing techniques that require computational techniques and computer technology that are not generally available.[1] This paper presents a land-use allocation model that utilizes accessible, efficient computer software and requires that objective criteria for allocation and important limiting conditions be characterized as deterministic linear relationships. A general land-use allocation model is developed and then a specific application of the model is presented in the paper. This application demonstrates how complex limitations constrain the rational objectives of decision-makers. The model is used to determine a mix of land uses that optimizes the objective criterion subject to these limiting constraints.

Post-optimality analysis of the constrained, optimal solution allows consideration of how uncertainty can affect the optimal land-use allocations. Post-optimality analysis examines: (1) the effects of changes in costs and profits due to deviations of important parameters from expected conditions, (2) the effects of changes in constraining relationships, and (3) opportunity costs associated with each constraint faced by the decision-maker. Conclusions on the model and potential applications of it are presented along with recommendations for further work in this area.

II. THE LAND-USE ALLOCATION MODEL

The developer's decision problem requires scarce resources (land, cash, lines of credit, etc.) to be allocated to the various land uses so that the value of the developer's utility function is optimized. In addition to resource constraints, most developments must meet societal constraints that reflect both economic and political considerations. This problem is a variation of the fundamental management problem in economics. In any economic system or activity the managerial problem requires that managers allocate scarce resources to competing economic activities in order to maximize (or minimize) the value of an appropriate objective function while satisfying technological, resource and societal constraints.

The Developer's Decision Model

A general functional representation of the developer's decision problem, which is a constrained optimization problem, is:

$$\text{Maximize (or Minimize):} \quad z = f(x_1, x_2, \ldots, x_m)$$

subject to:

$$g_i(x_1, x_2, \ldots, x_m) \quad i = 1, 2, \ldots, n$$

The function z represents the objective function; the g_i represents the constraints that are established by technology, society or resource levels; and the x_j's represent the decision variables. The objective function could represent goals such as aggregate utility, but monetary objective functions are normally used. The functional relations above can be deterministic or stochastic as well as linear or nonlinear, and the constraints can be either equalities or inequalities. Complex relationships can defy actual computations of solutions, but problems that contain only linear deterministic relations are readily and efficiently solved with the aid of standard programs on electronic computers. The model developed in this paper is based on linear deterministic relations.[2] It is applicable and robust for most applications as long as the relations can be reasonably approximated by a linear relation over the "relevant range."[3] Violations of this assumption would limit the validity of the solution and analyses generated with the model.

Specifically, the developer's decision problem requires the following:

1. Potential future cash flows associated with various alternative uses must be estimated.

2. Land-use allocations should result in a maximum aggregate net present values of the future cash flows from the selected uses.[4]

3. The allocation in (2) must satisfy the resource, technological and societal constraints that limit the prerogatives of the decision maker.

The method to meet these requirements and solve the land-use allocation problem is described in the proceeding sections.

The Objective Function

The objective function of the land-use allocation model is specified in terms of the net present values of future cash flows per use-unit so that the optimal allocation maximizes the aggregate net equity present value. A method for determining the contribution that each unit of land use i, C_i, is described in Appendix A.[5] Appendix A* also examines the

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*Because of their length, Appendices A, B, C and D are not included with this paper. Copies of these Appendices can be obtained from either author.

effects of some variables on the contribution coefficients for use. The ranges estimated for each C_i in this sensitivity analysis are used in the post-optimality analysis of the problem.

The objective function can be stated as:[6]

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$$

where

m = the number of possible uses,

n = the number of periods required for the development,

C_{ij} = the net equity residual value per unit (or contribution coefficient) of use i in period j (This value may be zero or negative for some uses, e.g., open space or parking areas.),

X_{ij} = the number of units of use i in period j.

After specifying the objective function, the relevant constraints are specified.

The Constraints

The general form for constraints is:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ijk} X_{ij} \quad * b_k \qquad k = 1, 2, \ldots, p.$$

New notations are:

p = the number of constraints,

a_{ijk} = the technology coefficient for use i in period j for constraint k,

b_k = the constant value for constraint k (The b_k's are re ferred to as right-hand side values of simply RHS value.),

$*$ is $\geq$, $\leq$ or $=$, depending on the nature of constraint k.

This general form for constraints is used for the remainder of this paper.

Several common types of constraints are specified in Appendix B for demonstration purposes.[7] Although the listing is not exhaustive of all types, it generally represents common development constraints. This concludes the expository material on the model specification; the nature of the solution and post-optimality analysis are considered in the following sections.

The Optimal Solution

The problem, once formulated, can readily be solved using standard computer software for solving linear programming (LP) problems. The solution (from the optimal simplex tableau) provides the following:[8]

1. Optimal levels for all decision variables. In the preceding formulation, land-use allocations (including open space requirements), period-by-period short-term loans, period-by-period cash balances, and the optimal value of the objective function are readily determined in the optimal tableau.

2. Per-unit opportunity cost (foregone profits per-unit) of each binding constraint. For constraints that are not exhausted, the solution establishes the amount of "slack" in that constraint at the optimal level.

3. Effect on the objective function of forcing into the solution land uses that are not included in the optimal solution. This is important for situations where the developer chooses subjectively to alter the objective solution.

In summary, when the solution is generated, additional information is also generated. The solution includes the numbers of units of each use that will maximize the aggregate equity residual value. The effect of relaxing or tightening the various constraints can be determined from the solution information, which also allows a decision maker to assess opportunity costs of deviations from the recommended use allocations.

Post-Optimality Analysis[9]

Robustness is an important consideration for any model, especially deterministic models. The effects of uncertainty on deterministic decision models and solutions should be explicitly examined. Linear programming results can readily be examined to determine the effect of variations from presumed relations in two distinct areas: (1) changes in contribution coeffecients (C's) can be examined with sensitivity analysis of the objective function, and (2) changes in the right-hand side (RHS) values of the constraints can be examined with sensitivity analysis of tightness.

Sensitivity analysis requires a ceteris paribus assumption. Simultaneous changes in the C's or RHS values complicate the analysis considerably.[10] With individual C and RHS value changes, sensitivity analysis provides ranges for changes that can occur without destroying the relations of other variables in the solution. Sensitivity analysis can also provide an evaluation of the objective function at the limits for the established ranges of changes. The computations for sensitivity analysis can be readily handled by most computers, but the software for these computations is not necessarily a part of standard packages.[11]

The case that follows complements the above exposition and should also contribute to a better understanding of precisely how the model can be applied to yield useful solutions to land-use allocation problems.

III. APPLYING THE MODEL: AN ILLUSTRATIVE CASE STUDY

Setting

The parcel of real estate under consideration is the former South Base of the U.S. Naval Air Technical Training Center, which consists of 450 acres located just south of the University of Oklahoma in Norman, Oklahoma. This area was selected as the case example because of the availability of site and market data.[12] The following discussion of the development planning for this site is intended only as an illustrative example of the land-use allocation model and does not represent the findings or recommendations of previous studies of the site. Assume that the 450 acres can be purchased for $2.5 million and that the program of hypothesized uses is considered as a Planned Unit Development with a 3-year development period.

The Objective Function

The land-use allocation should maximize the aggregate net present value of the investor's equity position. Six general land uses were considered as profit centers in this analysis:

X_{1j} = square feet of garden apartment floor space built in year j,

X_{2j} = square feet of townhouse floor space built in year j,

X_{3j} = square feet of single-family residential lot space developed in year j,

X_{4j} = square feet of commercial retail floor space built in year j,

X_{5j} = square feet of office floor space built in year j,

X_{6j} = square feet of medical office floor space built in year j.

Open space (both jointly used and single use open space) is also recognized explicitly in the objective function, as well as land that is donated to the city for public uses. Therefore:

X_{7j} = square feet of open space that can be used jointly by apartments and townhouse to meet zoning requirements in year j,

X_{8j} = square feet of open space that can be used jointly by medical and clerical office uses to meet zoning requirements in year j,

X_{9j} = square feet of open space required by commercial retail space uses to meet zoning requirements in year j,

X_{10j} = square feet of land donated to the city for public uses.

The costs of all other required uses are assigned to the appropriate land use profit center. Uses X_7 through X_{10} are assigned a contribution coefficient of -1.27 in the objective function which is the cost per square foot of open space.

The contribution coefficients, C, of use i in year 1, were calculated as the net equity present value per square foot of use i in year 1. For uses that were built or developed after year 1, the contribution coefficients were calculated as follows:

$$C_{ij} = \frac{C_{i1}}{(1+r)^{j-1}}$$

where

C_{ij} = contribution coefficient of use i in year j,

r = investor's required rate of return.

The complete objective function for all land uses with non-zero contribution coefficients is specified in Table 1 and the input data are summarized in Appendix D.

The sensitivity analysis of the contribution coefficients to changes in exogenous variables was also analyzed. The growth rate of the rental income and operating expenses was varied along with the interest rate on the permanent loan. Results of this analysis are summarized in Table 2 and are used in the post-optimality analysis. From Table 2 it is obvious that the townhouse development is profitable (i.e., shows a positive net equity value per square foot of floor space) only under the most optimistic conditions. Therefore, townhouse development is excluded from further consideration in the model.

Constraining Conditions for the Optimal Solution

Four general types of constraints are specified for this example: zoning constraints, market constraints, total use constraints and cash budget constraints. Environmental constraints, which would be quite important in localities where serious environmental limitations exist, are discussed in Appendix B. In this particular case, however, specific environmental constraints are not recognized by the local planning commission. If environmental limitations were proposed or imposed, the impact on the planned allocations could be determined by adding those constraints and solving the problem.

In Table 3 the objective function and each constraint are described. The analysis and methods for developing the coefficients in these constraints are presented in Appendix C.[13]

The Optimal Land-Use Allocation[14]

The optimal solution values, presented in Table 4, call for 456 single-family detached homes to be built in the first year, as well as 70,000 square feet of clerical office space and 30,000 square feet of medical office space. These use allocations also require adequate streets and parking to be built in the first year. Although no building is specified in the

TABLE 1

Objective Function Contribution Coefficients for the Case Analysis

$$L = \sum_{i=1}^{10} \sum_{j=1}^{3} C_{ij}X_{ij}$$

GARDEN APARTMENT

$C_{11}X_{11} = .3551\ X_{11}$

$C_{12}X_{12} = .328\ X_{12}$

$C_{13}X_{13} = .304\ X_{13}$

TOWNHOUSE

$C_{21}X_{21} = -.02885\ X_{21}$

$C_{22}X_{22} = -.026\ X_{22}$

$C_{23}X_{23} = -.024\ X_{23}$

SINGLE FAMILY

$C_{31}X_{31} = 1.367\ X_{31}$

$C_{32}X_{32} = 1.265\ X_{32}$

$C_{33}X_{33} = 1.172\ X_{33}$

COMMERCIAL RETAIL SPACE

$C_{41}X_{41} = 5.193\ X_{41}$

$C_{42}X_{42} = 4.808\ X_{42}$

$C_{43}X_{43} = 4.453\ X_{43}$

OFFICE SPACE

$C_{51}X_{51} = 2.3145\ X_{51}$

$C_{52}X_{52} = 2.143\ X_{52}$

$C_{53}X_{53} = 1.985\ X_{53}$

MEDICAL OFFICE SPACE

$C_{61}X_{61} = 1.235\ X_{61}$

$C_{62}X_{62} = 1.1435\ X_{62}$

$C_{63}X_{63} = 1.059\ X_{63}$

OPEN SPACE FOR APARTMENTS AND TOWNHOUSES

$C_{71}X_{71} = -.127\ X_{71}$

$C_{72}X_{72} = -.117\ X_{72}$

$C_{73}X_{73} = -.108\ X_{73}$

OPEN SPACE FOR OTHERS

$C_{81}X_{81} = -.127\ X_{81}$

$C_{82}X_{82} = -.117\ X_{82}$

$C_{83}X_{83} = -.108\ X_{83}$

OPEN SPACE FOR COMMERCIAL USE

$C_{91}X_{91} = -.127\ X_{91}$

$C_{92}X_{92} = -.117\ X_{92}$

$C_{93}X_{93} = -.108\ X_{93}$

DONATED PUBLIC SPACE

$C_{10,1}X_{10,1} = -.127\ X_{101}$

$C_{10,2}X_{10,2} = -.117\ X_{102}$

$C_{10,3}X_{10,3} = -.100\ X_{103}$

TABLE 2

Sensitivity Analysis of the Contribution Coefficients

Variable Values			Contribution Coefficients Values					
Rental Income Growth Rate	Operating Expense Growth Rate	Interest Rate on Loan	C_{11} Garden Apartments	C_{21} Townhouse	C_{31} Single Family	C_{41} Commercial Space	C_{51} Office Space	C_{61} Medical Space
-.03	.06	.11	-2.7667	-2.7667	.8035	2.301	-3.2934	-6.175
-.03	.06	.10	-2.1826	-2.3893	.8084	3.084	-2.4836	-5.106
-.03	.03	.11	-1.8944	-2.1259	.9756	2.602	-1.989	-4.449
-.03	.06	.09	-1.7723	-2.0173	.813	3.854	-1.685	-4.048
-.03	.03	.10	-1.4757	-1.74639	.9806	3.385	-1.1748	-3.372
0	.06	.11	-1.4159	-1.6296	1.0214	3.850	- .9857	-3.122
-.03	0	.11	-1.2811	-1.5669	1.1445	2.861	- .8521	-2.944
-.03	.03	.09	-1.0622	-1.3715	.9854	4.155	- .3706	-2.31
0	.06	.10	- .9947	-1.2478	1.026	4.633	- .167	-2.04
-.03	0	.10	- .8588	-1.1849	1.1494	3.644	- .3242	-1.862
0	.03	.11	- .6954	- .9756	1.1935	4.151	.3451	-1.364
0	.06	.09	- .5789	- .872	1.0312	5.403	.6395	- .975
-.03	0	.09	- .4424	- .8088	1.1543	4.415	.7747	- .796
0	.03	.10	- .2713	- .5934	1.1984	4.934	1.1653	- .281
0	0	.11	- .0691	- .4111	1.3624	4.410	2.1314	.152
.03	.06	.11	- .0127	- .2897	1.2736	5.646	1.7339	.469
0	.03	.09	.146	- .2172	1.2033	5.704	1.975	.784
0*	0*	.10*	.3549*	- .0289*	1.3673*	5.193*	2.314*	1.235*
.03	.06	.10	.4113	.0927	1.2485	6.429	2.5541	2.301
0	0	.09	.7721	.3473	1.3722	5.964	3.1216	2.301
.03	.03	.11	.7316	.3651	1.4157	5.947	3.0669	2.228
.03	.06	.09	.8285	.4687	1.2534	7.2	3.3612	2.617
.03	.03	.10	1.1376	.7473	1.4206	6.730	3.8871	3.311
.03	0	.11	1.3398	.9296	1.5846	6.207	4.216	3.745
.03	.03	.09	1.5549	1.1235	1.4255	7.5	4.6942	4.376
.03	0	.10	1.7638	1.3119	1.5895	6.99	5.0363	4.828
.03	0	.09	2.181	1.688	1.5944	7.760	5.8434	5.893

*Values used to solve for optimum land use allocation.

TABLE 3

Optimum Land-Use Allocation Problem Set-Up

MAXIMIZE: Z =

(Apartments)	$.355X_{11} + .328X_{12} + .304X_{13}$
(Single-Family Residences)	$+ 1.367X_{31} + 1.265X_{32} + 1.172X_{33}$
(Commercial Retail Space)	$+ 5.193X_{41} + 4.800X_{42} + 4.453X_{43}$
(Office Space)	$+ 2.3145X_{51} + 2.143X_{52} + 1.985X_{53}$
(Medical Office Space)	$+ 1.235X_{61} + 1.1435X_{62} + 1.059X_{63}$
(Open Space for Apartments)	$- .127X_{71} - .117X_{72} - .108X_{73}$
(Open Space for Medical and Office)	$- .127X_{81} - .117X_{82} - .108X_{83}$
(Open Space for Retail Space)	$- .127X_{91} - .117X_{92} - .108X_{93}$
(Donated Public Space)	$- .127X_{101} - .117X_{102} - .108X_{103}$

SUBJECT TO: (The constraints are categorized, numbered and briefly described by use topic.)

Zoning Constraints:

(1) Floor Area - X_1	$.5148X_{1j} - .4X_{7j} - .4X_{11j} \leq 0$
(2) Open Space - X_1	$1.8X_{1j} - X_{7j} \leq 0$
(3) Livability Space - X_1	$.95X_{1j} - X_{7j} + X_{11j} \leq 0$
(4) Recreational Space - X_1	$.13X_{1j} - .8X_{7j} + .8X_{11j} \leq 0$
(5) Total Cars - Zoning - X_1	$X_{1j} - .1763X_{11j} \leq 0$
(6) Total Cars - Market - X_1	$X_{1j} - .158X_{11j} = 0$
(7) Land Area Coverage - X_3	$.32X_{3j} - X_{12j} = 0$
(8) Floor Area - X_4	$.5X_{4j} - .5X_{9j} - .5X_{13j} \leq 0$
(9) Total Cars - X_4	$200X_{4j} - 270X_{13j} \leq 0$
(10) Floor Area - $X_5 + X_6$	$.5X_{5j} + .5X_{6j} - .3X_{8j} - X_{14j} \leq 0$
(11) Total Cars - $X_5 + X_6$	$300X_{5j} + 300X_{6j} - 270X_{14j} \leq 0$

Marketing Constraints

(12) Maximum Size - X_1	$\sum_{j=1}^{3} X_{1j} \leq 479,400$
(13) Minimum Size - X_3	$X_{31} \geq 1,672,000$
(14) Maximum Size - X_3	$\sum_{j=1}^{3} X_{3j} \leq 5,016,000$
(15) Timing - X_4	$X_{41} = 0$
(16) Timing - X_4	$X_{42} = 0$
(17) Maximum Size - X_4	$X_{43} \leq 75,000$
(18) Minimum Size - X_4	$X_{43} \geq 30,000$
(19) Maximum Size - X_5	$\sum_{j=1}^{3} X_{5j} \leq 70,000$
(20) Maximum Size - X_6	$\sum_{j=1}^{3} X_{6j} \leq 30,000$
(21) Minimum Size - $X_5 + X_6$	$\sum_{j=1}^{3} X_{5j} + X_{6j} \geq 20,000$
(22) Donated Space Required	$X_{10,3} \geq 43,560$

Total Land-Use Constraint

(23) Total Maximum Size	$\sum_{j=1}^{3} 1.213X_{1j} + \sum_{j=1}^{14} \sum_{j=1}^{3} X_{1j} \leq 19,602,000$

Cash Budget Constraints

(24) First Year

$$3.0564X_{11} - .5051X_{31} + 5.2497X_{51} + 10.7786X_{61} + .127X_{71} + .1278_1 \leq 2,500,000$$

(25) Second Year

$$2.642X_{11} + 3.0564X_{12} - .5051X_{31} - .5051X_{32} + 4.1401X_{51} + 5.2497X_{52} + 9.1912X_{61} + 10.7786X_{62} + .127X_{71} + .127X_{72} + .127X_{81} + .127X_{72} \leq 5,000,000$$

(26) Third Year

$$2.2465X_{11} + 2.6420X_{12} + 3.0564X_{13} - .5051X_{31} - .5051X_{32} - .5051X_{33} - .7146X_{43} + 3.0984X_{51} + 4.1401X_{52} + 5.2497X_{53} + 7.712X_{61} + 9.1912X_{62} + 10.7786X_{63} + 1.127X_{71} + .127X_{72} + .127X_{73} + .127X_{81} + .127X_{82} + .127X_{93} + .127X_{103} \leq 5,000,000$$

TABLE 4

Final Solution

Original Variables	Values	Slack Variables*	Values
$X_{1,1}$	= 0	X_s, Eq. 8	= 12,060
$X_{1,2}$	= 0	X_s, Eq. 10, year 1	= 61,110
$X_{1,3}$	= 0	X_s, Eq. 12	= 479,000
$X_{3,1}$	= 5,016,000	X_s, Eq. 13	= 3,344,000
$X_{3,2}$	= 0	X_s, Eq. 18	= 45,000
$X_{3,3}$	= 0	X_s, Eq. 21	= 80,000
$X_{4,3}$	= 75,000	X_s, Eq. 23	= 12,595,650
$X_{5,1}$	= 70,000	X_s, Eq. 24	= 4,342,740
$X_{5,2}$	= 0	X_s, Eq. 25	= 6,968,030
$X_{5,3}$	= 0	X_s, Eq. 26	= 7,133,390
$X_{6,1}$	= 30,000		
$X_{6,2}$	= 0	OBJECTIVE FUNCTION VALUE =	
$X_{6,3}$	= 0		
$X_{7,1}$	= 0	\$7,385,197	
$X_{7,2}$	= 0		
$X_{7,3}$	= 0		
$X_{8,1}$	= 0		
$X_{8,2}$	= 0		
$X_{8,3}$	= 0		
$X_{9,3}$	= 0		
$X_{10,3}$	= 43,560		
$X_{11,1}$	= 0		
$X_{11,2}$	= 0		
$X_{11,3}$	= 0		
$X_{12,1}$	= 1,605,120		
$X_{12,2}$	= 0		
$X_{12,3}$	= 0		
$X_{13,3}$	= 55,560		
$X_{14,1}$	= 111,110		
$X_{14,2}$	= 0		
$X_{15,3}$	= 0		

*X_s, p = slack variable for the p^{th} equation.

second year, 75,000 square feet of retail floor space is allocated in the third year along with the required parking space. The donated land is also dedicated in the third year, and one hundred and twenty (120) acres are left undeveloped. The allocation values could probably be anticipated, but the staging of these uses is not as obvious. More exact staging results are possible if smaller definitions of time intervals (months instead of years) are specified in the model.[15]

The SFR allocation and timing, X_{31}, is not surprising given the available financing for the SFR development. The timing is a result of the market specification that allows all SFR's to be sold in the first year. A more realistic set of market limitations would have changed this significantly. However, in the interest of clarity, the annual time periods and the market specifications used have not demonstrated fully the effects of these constraints on the timing of the project. This timing also dramatically affects the timing of the other uses in the project.

The slack variables that are presented in the solution, particularly the cash balances, are also of interest. These balances are an artifact of the preceding timing patterns. Cash balances for the three years are indicated at \$4,342,740 (X_s, Equation 24), \$6,968,030 (X_s, Equation 25) and \$7,133,390 (X_s, Equation 26). This result occurs because of the interaction of the marketing constraints and the cash flow patterns. The price schedule for the rental properties impacts on cash flow patterns and this set of cash balances. If the market price for garden apartments were higher, the cash balances would rapidly disappear as this use was brought into the solution.

Post-Optimality Analysis

Table 5 gives the results of the post-optimality sensitivity analysis of the contribution coefficient value, and Table 6 presents the sensitivity analysis of tightness. These tables show ranges for the coefficients and RHS values and the objective function results that correspond to those ranges. Values outside of these ranges would cause the optimal solution to fail. Table 5 can be compared with Table 2 (sensitivity analysis of the coefficients to changing parameters in their computations) to determine which parameter changes are compatible with the recommended optimal solution values. For example, the office space coefficient range is 2.143 to 767,554.3 in Table 5. Table 2 indicates that the contribution of the office space would be in this range for ordered triples of income growth rates, operating expense growth rates and interest rates on the order of: .03, .06, .10; .0, .0, 0.9; .03, .03, .11; etc., respectively. If these variables change enough for the coefficients to go out of the range in Table 5, the optimal land-use allocation would require a different solution.

The sensitivity of the optimum allocation and the objective function value to the tightness of the constraints, i.e., changes in the RHS values, is presented in Table 6 and analyzed in a manner similar to the contribution coefficient analysis. In this case, however, the comparison would be with the right-hand side values in Table 3. The marketing limitations are binding constraints for most uses. An examination of the sensitivity analysis indicates how much the objective function would change, given a loosening (or tightening) of the marketing constraints. For example, the effect of increasing the amount of the office space that could be sold (see Equation 19). The limit for this constraint would be 897,240 square feet. If this amount could be successfully sold, the objective function value would increase to

TABLE 5

Post-Optimality Sensitivity Analysis
Of the Contribution Coefficients

Variable	Original Coefficient (00.0000)	Range of Coefficients Lower Limit	Range of Coefficients Upper Limit	Range of Objective Function Values Lower Limit	Range of Objective Function Values Upper Limit
$X_{1,1}$	0.3550	*	1.7489	73.8520	73.8520
$X_{1,2}$	0.3280	-981464.6250	1.6112	73.8520	73.8520
$X_{1,3}$	0.3040	-905967.1875	1.4872	73.8520	73.8520
$X_{3,1}$	1.3670	1.2650	+INFINITY	68.7357	NONE
$X_{3,2}$	1.2650	-INFINITY	1.3670	NON-BASIC VARIABLE. . .	.NO EFFECT
$X_{3,3}$	1.1720	-INFINITY	1.3670	NONE	73.8520
$X_{4,3}$	4.4530	0.0000	+INFINITY	70.5122	NONE
$X_{5,1}$	2.3145	2.1430	767554.3125	73.7319	537360.0000
$X_{5,2}$	2.1430	-INFINITY	2.3145	NON-BASIC VARIABLE. . .	.NO EFFECT
$X_{5,3}$	1.9850	-INFINITY	2.3145	NON-BASIC VARIABLE. . .	.NO EFFECT
$X_{6,1}$	1.2350	1.1435	+INFINITY	73.8245	NONE
$X_{6,2}$	1.1435	-INFINITY	1.2350	NON-BASIC VARIABLE. . .	.NO EFFECT
$X_{7,1}$	-0.1270	-INFINITY	-0.0000	NONE	73.8520
$X_{7,2}$	-0.1170	-INFINITY	-0.0000	NONE	73.8520
$X_{7,3}$	-0.1080	-INFINITY	-0.0000	NONE	73.8520
$X_{8,1}$	-0.1270	-133169.0000	0.0000	73.8520	73.8520
$X_{8,2}$	-0.1170	-122683.1875	0.0000	73.8520	73.8520
$X_{8,3}$	-0.1080	-113246.0000	0.0000	73.8520	73.8520
$X_{9,3}$	-0.1080	-INFINITY	0.0	NON-BASIC VARIABLE. . .	.NO EFFECT
$X_{10,3}$	-0.1080	-INFINITY	0.0	NONE	73.8990
$X_{11,1}$	0.0	-133168.8750	0.2202	73.8520	73.8520
$X_{11,2}$	0.0	-122683.1250	0.2027	73.8520	73.8520
$X_{11,3}$	0.0	-113245.9375	0.1870	73.8520	73.8520
$X_{12,1}$	0.0	-0.3187	+INFINITY	68.7357	NONE
$X_{12,2}$	0.0	*	0.3187	66.5395	73.8520
$X_{12,3}$	0.0	-INFINITY	0.6094	NON-BASIC VARIABLE. . .	.NO EFFECT
$X_{13,3}$	0.0	-6.0115	0.0000	70.5122	73.8520
$X_{14,1}$	0.0	-0.0823	0.0000	73.7605	73.8520
$X_{14,2}$	0.0	-INFINITY	0.0	NON-BASIC VARIABLE. . .	.NO EFFECT
$X_{14,3}$	0.0	-INFINITY	0.0	NON-BASIC VARIABLE. . .	.NO EFFECT

*Number is less than -9,999,999.

TABLE 6

Sensitivity Analysis of Tightness

Equation	Original Constraint (00,000)	Range of Value: Lower Limit (00,000)	Range of Value: Upper Limit (00,000)	Range of Objective Function Values: Lower Limit (00,000)	Range of Objective Function Values: Upper Limit (00,000)
Eq. 1, yr. 1	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 1, yr. 2	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 1, yr. 3	0.0	0.0	+INFINITY	73,8520	NONE
Eq. 2, yr. 1	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 2, yr. 2	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 2, yr. 3	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 3, yr. 1	0.0	0.0	*	73.8520	5,103,103.0000
Eq. 3, yr. 2	0.0	0.0	*	73.8520	4,701,290.0000
Eq. 3, yr. 3	0.0	0.0	*	73.8520	4,339,657.0000
Eq. 4, yr. 1	0.0	0.0	0.9777	73.8520	74.0072
Eq. 4, yr. 2	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 4, yr. 3	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 5, yr. 1	0.0	0.0	0.0220	73.8520	74.1144
Eq. 5, yr. 2	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 5, yr. 3	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 6, yr. 1	0.0	0.0	0.5020	73.8520	67.9823
Eq. 6, yr. 2	0.0	0.0	0.5020	73.8520	68.4491
Eq. 6, yr. 3	0.0	0.0	0.5020	73.8520	68.8706
Eq. 7, yr. 1	0.0	0.0	16.0512	73.8520	73.8520
Eq. 7, yr. 2	0.0	0.0	0.0000	73.8520	73.8520
Eq. 7, yr. 3	0.0	0.0	10.7008	73.8520	67.3312
Eq. 8, yr. 3	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 9, yr. 3	0.0	0.0	65.1120	73.8520	73.8520
Eq. 10, yr. 1	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 10, yr. 2	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 10, yr. 3	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 11, yr. 1	0.0	0.0	164.9999	73.8[illegible]0	73.8520
Eq. 11, yr. 2	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 11, yr. 3	0.0	0.0	+INFINITY	73.8520	NONE
Eq. 12	4.7900	0.0	+INFINITY	73.8520	NONE
Eq. 13	16.7200	-INFINITY	50.1600	NONE	73.8520
Eq. 14	50.1600	16.7200	145.5816	28.1395	204.2933
Eq. 17	0.7500	0.3000	1.6802	71.8481	77.9940
Eq. 18	0.3000	-INFINITY	0.7500	NONE	73.8520
Eq. 19	0.7000	0.0000	5.9724	72.2318	92.9984
Eq. 20	0.3000	0.0	4.3290	73.4815	78.8278
Eq. 21	0.2000	-INFINITY	1.000	NONE	73.8520
Eq. 22	0.4356	0.1944	126.3921	73.8780	60.2487
Eq. 23	196.0200	70.0635	+INFINITY	73.8520	NONE
Eq. 24	25.0000	0.0	+INFINITY	73.8520	NONE
Eq. 25	50.0000	0.0	+INFINITY	73.8520	NONE
Eq. 26	50.0000	0.0	+INFINITY	73.8520	NONE

*Number is greater than 99,999,999.

$9,299,840. Similar examinations of the other ranges indicate the effect of the marketing limitations on the objective function. The shadow prices for these constraints also provide some insight into the effect of changes in marketing limitations.

The limitations that were placed on this application limit the extent of the meaningfulness of the sensitivity analysis. A more complicated application would yield more meaningful analysis, especially with regard to the cash budget limitations and the timing of the uses in the development. Work is continuing on a more realistic and complicated application of the model.

IV. CONCLUSION

The allocation of a parcel of land to a mixture of nonexclusive uses is an old and difficult problem. Linear programming is not a new decision aid; however, using it to solve such a problem is a relatively new application. It also is a logical progression of computer applications to real estate problems, given that inputs for the solution are generated by computer-oriented discounted cash flow models.

The validity of the decision model for optimum land use allocations is limited by the assumptions of the linear programming algorithm, the ability of the decision-maker to specify his objectives, the limiting conditions in the linear programming framework, and the availability of efficient computer software. While recognizing the underlying assumptions of linear programming, this paper demonstrates how decision-makers can use this technique, first on an abstract level and then on a practical level.

The usefulness of the optimal solution for planning purposes is apparent. In addition, the paper points out that the post-optimality analysis generates an array of rich information for the decision-maker. The sensitivity of the optimum allocation to errors of estimation, changes in the states of nature, and opportunity cost can be considered in the post-optimality analysis.

Recommendations for Further Research in this Area

Once the land-use allocation problem has been set up, it can be used as a tool to evaluate proposed government policies and regulations. Environmental limitation and the impact of environmental regulations on land use allocation can be studied in the context of the proposed model. The effect that existing zoning controls have on land-use allocations is also evident, as well as how much it would be worth to the decision-maker to have the zoning regulations changed.

One recommendation for extended research in this area is the further refinement and sophistication of the existing model so that it can better represent the problem faced by the decision-maker. This would mean including more realisitc time periods, differing levels of marginal revenue and costs, and explicit measures of risk in the model. Research is also needed on the application of the model to the determination of the optimum mix of uses in smaller types of real estate projects, for example, determining the optimum mix of retail floor space in a shopping center or the optimum mix of apartment types, recreation facilities, and open space in an apartment complex. Finally, as noted earlier the investigation of public policy with the model should have worthwhile results.

The value of this type of quantitative model to the practitioner is obvious but this value is likely to be regarded as apocryphal, at best, by practitioners. This is primarily because of the complexity of the mathematics underlying the model and the lack of easily readable explanations and adaptations of the model. Undoubtedly, the model as presented in its basic form in this paper will not convince a skeptical practitioner of its worth. However, if it is employed successfully to analyze several large real estate investments, its usefulness may become more obvious. Indeed, some day this decision model for land-use allocations may be as common as its predecessor, the discounted cash flow model.

FOOTNOTES

[1] See for example Noel M. Edelson, "The Developer's Problem, or How to Divide a Piece of Land Most Profitably," Journal of Urban Economics, 2 (1975), pp. 349-365.

[2] Interested readers will find some of the alternative formulations and solution techniques presented in H. M. Wagner, Principles of Operations Research (New York: Prentice-Hall, 1969) and Claude McMillan, Jr., Mathematical Programming (New York: John Wiley & Sons, Inc., 1970). Some of these alternatives include quadratic programming, geometric programming, integer programming, goal programming and stochastic programming.

[3] Robust models are models that are sensitive to violations of the assumptions of the model.

[4] This criterion ignores the risk dimensions of the problem and the risk-averse nature of most persons. The required rate of return (discount rate) could be (subjectively) adjusted to crudely reflect the riskness of cash flows for each use. We are currently attempting to develop a model that allows recognition of the stochastic nature of future cash flows as well as an explicit method for accommodating risk considerations into an allocation model.

[5] As mentioned previously, Appendix A presents the detailed computations that are required to estimate the C's. Briefly, each C_i can be estimated as follows:

If $$NCV_{i,h} = \sum_{t=1}^{n} \frac{NCF_{it}}{(1+r)^t} + \frac{ER_{in}}{(1+r)^n} - \sum_{t=1}^{n} \frac{RE_{it}}{(1+r)^t}$$

where

i = an index of the proposed land uses

h = the hypothesized size of project use i in spatial units (e.g., acres, square feet)

n = the number of periods project i is held

t = an index of time

r = investor's required rate of return on equity

NCF_{it} = net cash flow in period t from use i

ER_{in} = equity reversion value at the end of the holding period (period n) from use i

RE_{it} = required equity amount in period t for use i

then

$$C_i = \frac{NEV_{i,h}}{h}$$

So C_i is a measure of the discounted net future cash flows that accrue to the equity investor per spatial unit of use i.

[6]The formula and discussion assume that the project under consideration requires time-staging (i.e., the development of the project continues over more than one period of time rather than being accomplished instantaneously). This formulation requires a second subscript for uses and coefficients to represent the time period when the uses are developed.

[7]As noted, Appendix B details the derivations for several common types of constraints. A brief presentation of two marketing constraints is given here to demonstrate the usual considerations in developing constraints.

Let q_{ij} represent the maximum quantity of units of use i for period j and let $M_{ij} = \sum_{t=0}^{j} q_{ij}$. Assume that the relevant range for use i (i.e., the quantity range over which marginal costs and marginal revenues are constant for use i) is bounded by L_{im} below and M_{im} above (e.g., $L_{im} \leq \sum_{j=1}^{m} X_{ij} \leq M_{im}$).

The marketing constraints for use i would be represented by the following M+1 constraints:

$$
\begin{array}{ll}
X_{i1} & \leq q_{i1} \\
X_{i1} + X_{i2} & \leq M_{i2} \ (M_{i2} = q_{i1} + q_{i2}) \\
X_{i1} + X_{i2} + X_{i3} & \leq M_{i3} \ (M_{i3} = q_{i1} + q_{i2} + q_{i3}) \\
\cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot & \cdot \ \cdot \\
\cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot & \cdot \ \cdot \\
\cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot \ \ \cdot \ \cdot & \cdot \ \cdot \\
X_{i1} + X_{i2} + X_{i3} + \ldots \ \ \ldots + X_{im} & \leq M_{im} \ (M_{im} = \sum_{j=1}^{M} q_{ij}) \\
X_{i1} + X_{i2} + X_{i3} + \ldots \ \ \ldots + X_{im} & \geq L_{im}.
\end{array}
$$

If time staging is not required for marketing use i, only the last two constraints would be needed.

[8]See the previous citations on mathematical programming for a discussion of the simplex method for solving LP problems. These sources also discuss the computations for sensitivity analysis and resource analysis that are presented in the following section.

[9]The mathematics of the simplex algorithm are intrinsic to the computations for post-optimality analysis. Assuming that most readers are not familiar with this algorithm, this section is limited to descriptive comments.

[10]Simultaneous changes require an analysis of systems of vector inequalities with multiple unknown change quantities. This generally results in an infinite number of solutions for the change quantities.

[11]For interested readers that do not have access to such software, the authors have a reasonably efficient Fortran program that determines an optimal solution using the simplex algorithm and also provides a complete printout of the above sensitivity analyses, including ranges and objective function evaluation for the ranges. A listing of this program may be obtained by writing Daniel B. Kohlhepp, Division of Finance, College of Business Administration, University of Oklahoma, Norman, Oklahoma 73019.

[12]This area was considered as a site for a Planned Unit Development in a 1969 study, Bishop Creek Village, by Environics Associates of Houston, Texas. More recently, alternative land uses for the site were evaluated by Daniel Kohlhepp and Rodney Evans, Feasibility Study of the South Base, Center for Economic and Management Research, College of Business Administration, University of Oklahoma, 1974.

[13]The development of the constraints for the Livability Space Ration (LSR) for garden apartments. FHA standards require that livability space, defined as open space not including paved areas, be at least 95 percent of the floor area of garden apartments.

If

X_{ij} = number of square feet of garden apartments built in year j

X_{7j} = open space for garden apartments (including uncovered parking areas) in square feet, developed in year j

X_{11j} = uncovered parking area for garden apartments in square feet, built in year j.

The LSR requirement states that

$$\frac{\text{Livability Space}}{\text{Floor Area}} \geq .95$$

or in the preceding symbols

$$\frac{X_{7j} = X_{11j}}{X_{1j}} \geq .95 \text{ or } .95X_{ij} - X_{7j} + X_{11j} \geq 0.$$

[14]Several computational problems which are indigenous to the application of linear programming algorithms to land use decisions should be noted. The large number of variables and equations necessary for this type of problem often exceed the capacity of canned programs. (This simple illustration had 78 variables and 78 equations.) The dimension statements must be modified for the main program and each subroutine. Large problems of this nature also require substantial amount of input data. A unique subroutine allows the complete specification of the initial tableau while only directly specifying the values of the non-zero cells in the original problem, the right hand side values, and the type of constraint. Another problem which is common in this type of application is the underflows and overflows caused by the wide range of values in the constraints. This problem was overcome by scaling down the right hand side values by 10^5.

[15]The planning benefits of smaller units of time are not clearly demonstrated in this simple example. A more complex set of cash constraints would be needed, but the solution would provide invaluable information for planning and staging the development. Such constraints allow a determination of the optimal period-by-period levels of lines of credit, optimal period-by-period cash balances as well as period-by-period levels of accounts payable, construction loans, inventories by stages, etc.

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APPENDIX A

Net Present Value Coefficients

In order to analyze net present value coefficients, each feasible alternative use must be identified. The uses reflect profit (or loss) centers and are expressed in terms of some spatial unit; for example, garden apartments can be considered as a possible use. Revenues and expenses (for constructing and operating garden apartments) must be specified. The appropriate spatial dimension of this use may be the square footage of floor area. The contribution of a use to the total net present equity value of the project must be determined as a second step. This net present equity value contribution must be expressed in the same spatial dimension as the possible use. Therefore, in its simplest form the objective function can be expressed as:

Maximize:

$$NPEV = \sum_{i=1}^{n} C_i X_i \tag{1}$$

where

NPEV = net present equity value of the entire project, in dollars,

X_i = amount of space of use i developed, i.e., X_i = the total number of square feet of garden apartments in the project,

C_i = net present equity value contribution in dollars per unit of use i; i.e., C_1 = the net present equity value contribution of one square foot of garden apartment floor space.

A check verifies the dimension consistency of the expression (i.e., Max:

$\$NPEV = (\frac{\$i}{xi})\ (X_i) = \$_i$.

After spatial dimensions of each use have been determined, a "typical" project size of this use is hypothesized. Given the size of the development, the total cost to develop the use is estimated. Development costs must include the land acquisition and planning costs, building and site improvement costs, required off-site improvement costs and construction financing expenses. From this total cost figure, the estimated permanent, long-term loan amount is deducted to calculate the initial required equity of the particular use, so that

$$E_{oi} = TC_i - L_i \tag{2}$$

where

E_{oi} = original required equity amount for use i ,

TC_i = total construction costs incidental to use i,

L_i = loan amount of the permanent financing available for use i.

It is important to recognize that each use i is a profit center, and net present value contributions for each use must be calculated. Therefore, even though the use under consideration is expressed as the square footage of garden apartment floor space, costs to develop that square footage must reflect the land cost for the total covered area, the recreational facilities required to meet competitive standards and effect required improvements such as streets and parking spaces.

The present value of the expected after-tax cash flow must be generated for each use in the project. This can be done using the following algorithm:[1]

$$TPVE_i = \sum_{j=1}^{n} \frac{(GI_i)\ (1+g_i)^{j-1}\ (OC_i) - OE_i\ (1+h)^{j-1} - I_{ij} - A_{ij} - T_{ij}}{(1+r)^j} + \frac{Pn - UM_n - GT_n}{(1+r)^n} \tag{3}$$

where

$TPVE_i$ = total present value of the equity capital for a project of use i ,

GI_i = gross income generated in year 1 of use i ,

g_i = annual growth rate of the gross income for use i,

OC_i = expected occupancy rate of the gross income for use i,

OE_i = operating expenses in year 1 of use i,

h_i = annual growth rate of operating expenses of use i,

I_{ij} = interest payable in year j for use i,

A_{ij} = required debt amortization in period j for use i,

T_{ij} = taxes payable in period j for use i,

r = investor's required rate of return on equity,

P_n = selling price of the property in period n,

UM_n = unpaid mortgage balance payable in year n,

GT_n = capital gains tax payable in period n.

The net present equity value of use i ($NPVE_i$) is found by subtracting the original required equity amount (E_{oi}) from the estimated total present value of the equity for use i ($TPVE_i$).

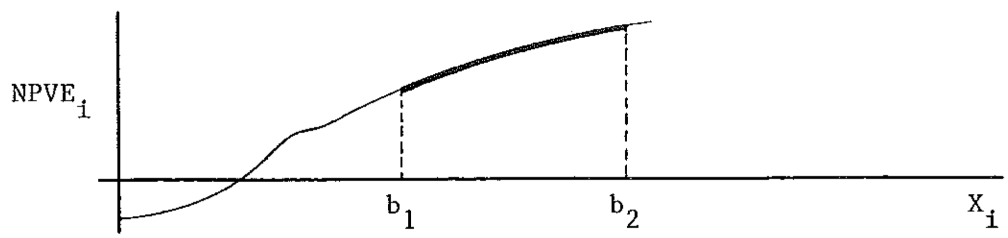
$$NPVE_i = TPVE_i - E_{oi} \tag{4}$$

The equity contribution of use i (C_i) is simply the $NPVE_i$ divided by the amount of space required for a project of the hypothesized size (HS_i). Therefore,

$$C_i = \frac{(NPVE_i)}{(HS_i)} \tag{5}$$

The important assumption involved here is that the marginal equity contribution (C_i) remains constant over the relevant range. That is, the amount of space allocated to use i in the final analysis will be within some range over which the marginal contribution is constant and which the hypothesized development is representative. Figure A-1 is a graphical representation of this assumption.

FIGURE A-1



$$\frac{NPVE}{X_i} = \text{Constant } C_i$$

The amount of space allocated to use i must be limited by the relevant range in the analysis. This is done in the technological constraints which are stated to require $b_1 \leq X_i \leq b_2$. The post-optimality analysis evaluates the effects of this assumption on the results.

The sensitivity of C_i to errors in estimation, changes in projected states of nature, or changes in input value should be considered. Sensitivities can readily be estimated by using a discounted cash flow model. The decision-maker can select the variables that are most susceptible to estimation errors or general variability and specify an optimistic estimate and a pessimistic estimate for the variables. The effects of these changes on C_i should be considered separately and jointly with possible changes in the other variables. This procedure provides a range of C_i's that vary from the worst set of circumstances to the best set of circumstances surrounding the project development. These values, used in post-optimality analysis, allow a decision-maker to specifically evaluate the effect that uncertain outcomes (i.e., changes in interest rates, revenue growth, etc.) have on the objective function. In this indirect fashion, the model allows a consideration of uncertainty.

Up to this point, only "profit center space uses" have been considered for the objective function. Some required space uses are not "profit center uses" but must also be recognized in the model (for example: open space, paved space and covered land space).

The costs and benefits to the aggregate net present value of equity for these required space uses are accounted for in the determination of the C's for the profit center space uses. Therefore, although space is allocated to these uses, the C_i for these uses is zero. The objective function can be stated as follows:

$$\text{MAX:} \quad \text{NPVE} = C_1X_1 + C_2X_2 + \ldots + C_iX_i + C_jX_j + \ldots + C_nX_n \quad (6)$$

where

X_1 to X_i = profit center space uses,

C_1 to $C_i \neq 0$,

X_j to X_n = required space uses,

C_j to $C_n = 0$.

Another implicit assumption in the estimation of the objective function is that the C_i's are independent of each other; that is, the value of C_i for use X_1 is not significantly related to the value of, or amount of, space allocated to use $X_2 \ldots X_n$. Given that each use is required to be within "relevant ranges," the assumption does not seem unreasonable.

FOOTNOTE

[1]This model can abe solved with any discounted-cash flow (DCF) program. Several DCF programs have been adapted to real estate development problems. Some of the better known adaptations include those due to Professors Paul Wendt, James Graaskamp, James Cooper and Stephen Pyhr.

APPENDIX B
The Constraints

This appendix presents detailed derivations of the general form of several types of constraints. The types covered include total use constraints, market constraints, zoning limitations and cash budget considerations.

Total Use Constraint

A total use constraint assures that the total assigned uses do not exceed the total available land. For total use constraints the a_{ij} values represent the amount of land required for one unit of use i (in the selected square measure) and the RHS value is the total amount of land (in like measure) in the parcel. Required uses such as paved area and open space may or may not show up as separate uses. These alternatives are examined with the zoning requirements in another section.

Market Constraints

Market constraints are straightforward for most situations. This presupposes a constant marginal revenue over the "relevant" market range for each use. Under this assumption each use will have an upper and lower bound. The market limitations must take into account the staging of the development by periods, however. For example, if the maximum aggregate amount (over time, up to period j) for use i in period j is M_{ij}, the market constraints for use i would be:

$$
\begin{array}{ll}
X_{i1} & \leq M_{i1} \\
X_{i1} + X_{i2} & \leq M_{i2} \\
\cdot \quad \cdot & \cdot \\
\cdot \quad \cdot & \cdot \\
\cdot \quad \cdot & \cdot \\
X_{i1} + X_{i2} \ldots\ldots\ldots\ldots & X_{in} \leq M_{in} \\
X_{i1} + X_{i2} \ldots\ldots\ldots\ldots & X_{in} \geq LB_i
\end{array}
\tag{1}
$$

LB_i is a lower bound for the number of units of use i. LB_i could be the amount required to maintain the "integrity" of the PUD; or it could be the lower end of the constant returns-to-scale for the estimated C_i.

Zoning Requirements

Constraints that represent strict quantity limitations for certain types of uses are the simplest zoning constraints. Such constraints are analogous to the marketing constraints above. Under some zoning restrictions, proportions for the various types of uses must be specified along with given open space and parking space requirements. The constraints for such limitations are more complex.

A proportional use requirement could be stated as:

$$d \cdot q_1 \cdot x_1 \geq e \cdot q_2 \cdot x_2 \tag{2}$$

where

d:e = the ratio of use 1 to use 2,

q_1 = the quantity of area in a unit of use 1 (a_{ijk} from the total constraint),

q_2 = the quantity of area in a unit of use 2 (a_{zjk} from the total constraint).

Standard form for these constraints would be:

$$-d \cdot q_i \sum_{j=1}^{n} X_{ij} + e \cdot q_{i'} \sum_{j=1}^{n} X_{ij'} \leq 0 \qquad (3)$$

for as many pairs (or even sets) of uses i and i´ as are mentioned in the zoning requirements.

Paved space (streets, sidewalks and driveways) requirements in zoning rules can be handled by incorporating the cost of the required paved areas directly into the C_i's by including all required land area as a part of space requirements for the use. As an alternative, paved areas can be specified as separate uses with negative contributions and appropriate use proportion constraints would be specified for all other uses. This approach is more cumbersome, however, and adds little to the analysis. The latter (separate use) approach is well suited to handling open space requirements for PUD's, however. The contribution coefficients for open space are the present value of the cost of improvements required of open spaces.

The "sharing" of open space by various uses is commonly allowed by zoning rules. If open space simultaneously meets requirements for two or more uses under the zoning rules, the jointly-used open space must be designated as a separate use. The constraint statement for the preceding conditions is:

$$e \cdot q_1 \cdot \sum_{j=1}^{n} X_{ij} - d\,(X_{os} + X_{oi}) \leq 0 \qquad (4)$$

The new symbols are:

e and d, where the required ratio of open space to use i is d:e ,

X_{os} = the area of open space "shared" by use i and other uses ,

X_{oi} = the area of open space required exclusively for use i.

A similar specification can be used for other uses that allow jointly-required space.

Cash Budget Requirements[1]

If the analysis is used to establish the fair market value of a parcel, cash budget limitations need not be considered. Even in that context, it is probably not realistic to assume that developers have unlimited funds for the development of a parcel. Personal or corporate funds that can be committed to any single project or parcel are usually limited. Although the limit may be large enough to accommodate any use allocation, it should be considered explicitly to determine whether or not the cash budget constrains the use allocation.

Cash constraints rely on the following basic accounting identity:

(Ending cash balance) = (Beginning cash balance) + (Cash receipts) - (Cash expenditures)

Examples of cash receipts include sales receipts, payments on receivables, loans or other sources of cash. Cash expenditures include cash expenses, loan repayments and other cash disbursements. The preceding identity and the requirement that the ending balance for one period be the beginning balance for the next period establishes the appropriate specification of the cash budget constraints. This could describe an indefinite horizon for a recursive relation, but most developments have limited time horizons that can be divided into manageable discrete units of time for modeling purposes.

For expository and practical considerations, three one-year periods are used for the cash constraints, i.e., j = 3. The first period constraint should require:

$$BC + \sum_{j=1}^{3} \sum_{i=1}^{m} CR_{1ij}X_{ij} - \sum_{j=1}^{3} \sum_{i=1}^{m} CE_{1ij}X_{ij} + L_1 = S_1 \qquad (5)$$

or (collecting and rearranging)

$$BC + \sum_{j=1}^{3} \sum_{i=1}^{m} (CR_{1ij} - CE_{1ij})\ X_{ij} + L_1 - S_1 = 0 \qquad (6)$$

The new notation is:

BC_1 = beginning cash balance for period 1,

CR_{1ij} = cash receipts in period 1 for each unit of use i to be completed in year j,

CE_{1ij} = cash expenditures in year 1 for use i in year j,

L_1 = amount of borrowed funds in year 1 (the contribution coefficient for L_1 in the objective function would carry a negative sign at the level of interest paid on short-term borrowings; if borrowing is not required, L_1 will be zero),

S_1 = ending cash balance for year 1 (this may be zero but is not non-negative).

The constraint for year 2 would be:

$$\{BC + \sum_{j=1}^{3} \sum_{i=1}^{m} (CR_{1ij} - CE_{1ij})\ X_{ij} + L_1\} +$$

$$\sum_{j=1}^{3} \sum_{i=1}^{m} (CR_{2ij} - CE_{2ij})\ X_{ij} - (1+r)\ L_1 + L_2 + S_1 = 0 \qquad (7)$$

This constraint requires repayment of the loan for year 1 with interest due (r) but if necessary, a new loan, L_2, can be made. The beginning balance for period 2 is the ending balance, S_1 above (the preceding expression in brackets is equal to S_1). The expression representing the cash constraints for the third year is determined in an analogous fashion and the beginning balance for each period includes every entry on the left-hand side of the previous period constraint (except S_{j-1}). This progression of variables in a constraint generates cumbersome constraints and variable sets after several periods. In general, higher values for the objective function can be obtained by finer subdivisions of the time periods, i.e., months instead of years. However, the multiplicative growth variables and constraints is a mitigating consideration in further subdividing periods. In view of this, the cash budget presentation in this paper considers yearly cash budgets rather than monthly cash budgets.

Although this concludes the expository material on constraints, the types of constraints presented by environmental impact requirements can also be included as constraints. The authors are currently attempting to include some stochastic processes while keeping the same basic form and solution techniques for the model.

FOOTNOTE

[1]The basic approach to modeling the cash budget constraints is presented in James C. T. Mao, Quantitative Analysis for Financial Decisions (New York: The MacMillian Company, 1969), Chapter 5. This approach is a variation of the "warehousing" problem presented much earlier by A. Charnes, W. W. Cooper and M. H. Miller in "Application of Linear Programming to Financial Budgeting and the Costing of Funds," The Journal of Business, Vol. 32 (January 1959), pp. 20-46.

APPENDIX C

Computations for the Application

The derivations of an example of each type of constraint are provided in this Appendix.[1]

Zoning Constraints

Since the City of Norman does not have a Planned Unit Development District specified in its zoning code, each possible land use must meet the conditions for its appropriate zoning district. For example, the garden apartment must meet the requirements for the high-density, multifamily district (RM-6), etc.

The zoning requirements for the garden apartments are primarily specified by land use intensity measures. These are a series of ratios that attempt to relate the amount of floor space to other required land uses. The functional relationships of these ratios are summarized in Exhibit C-1.

The first restriction for garden apartments is the Floor Area Ratio (FAR), which is used to illustrate the computations required of a typical zoning constraint. The two new variables are:

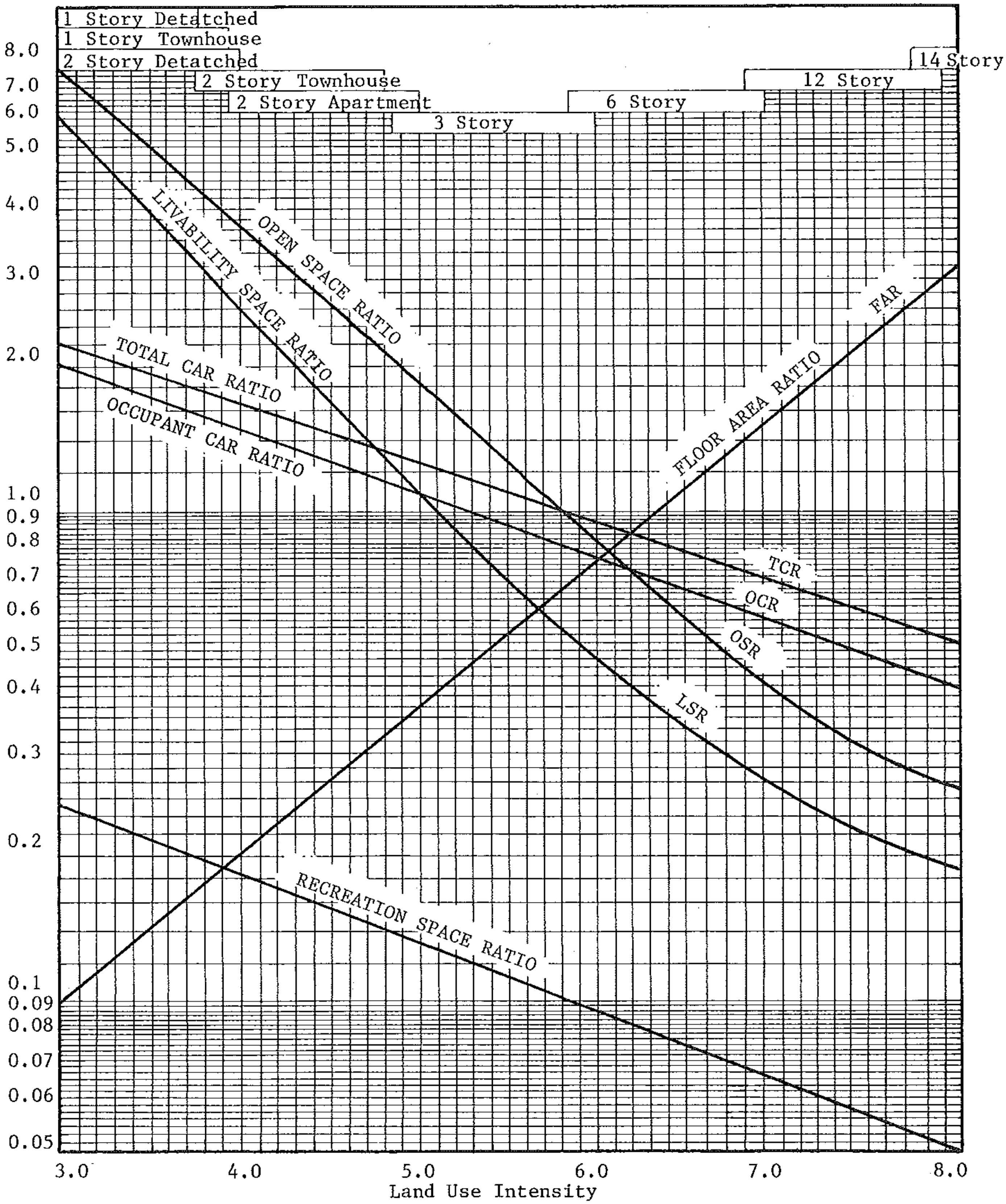
X_{gj} = square feet of ground area covered by buildings per square foot of garden apartment floor space. From typical construction plans, it is estimated that $X_{gj} = 1.213\ X_{1j}$,

X_{11j} = square feet of uncovered parking area required per square foot of garden apartment floor space.

The required FAR for garden apartments is:

$$\frac{\text{floor area in year } j}{\text{land area in year } j} = \leq .4.$$

EXHIBIT C-1



Source: Minimum Property Standards for Multi-Family Housing, Appendix B. Federal Housing Administration, Washington, D.C.: November 1963, p. 38.

Substituting from the preceding variables:

$$\frac{X_{1j}}{X_{gj} + X_{7j} + X_{11j}} = \frac{X_{1j}}{1.213X_{7j} + X_{11j}} \leq .4.$$

Rearranging terms, the FAR for garden apartments can be stated:

$$.5148X_{1j} - .4X_{7j} - .4X_{11j} \leq 0. \qquad (1)$$

Market Constraints[2]

The market constraints must reflect the maximum market demand for each use during each year of development or the total demand for the use over the development period. The market constraint should recognize a minimum development size if such a threshold exists.

The demand estimate for single-family housing space in the subject area was based on the existing supply of housing in Norman, the single-family to multifamily housing unit ratio, the expected population increase over the development period as well as the perceived competitive position of the subject development to capture the increased demand. In this manner, a market constraint for single-family housing was derived.

The market conditions for single-family housing required both a first year construction minimum as well as a total 3-year maximum.

$$X_{31} \geq 1,672,000 \qquad (2)$$

$$\sum_{j=1}^{3} X_{3j} \leq 5,016,000 \qquad (3)$$

Total Land-Use Constraints

The total land-use constraint is simple but important. This constraint guarantees that the total land-use allocation does not exceed the available land. Since, in this example, 450 acres are available for use, the total allocation must be equal to or less than 19,602,000 square feet as per Equation 4.

$$\sum_{j=1}^{3} 1.213X_{1j} + \sum_{j=1}^{3} X_{3j} + \sum_{j=1}^{3} X_{4j} + \sum_{j=1}^{3} X_{5j} + \sum_{j=1}^{3} X_{6j} +$$

$$\sum_{j=1}^{3} X_{7j} + \sum_{j=1}^{3} X_{8j} + \sum_{j=1}^{3} X_{9j} + \sum_{j=1}^{3} X_{10j} + \sum_{j=1}^{3} X_{11j} +$$

$$\sum_{j=1}^{3} X_{12j} + \sum_{j=1}^{3} X_{13j} + \sum_{j=1}^{3} X_{14j} \leq 19,602,000$$

or

$$\sum_{j=1}^{3} 1.213X_{1j} + \sum_{j=3}^{14} \sum_{j=1}^{3} X_{1j} \leq 19,602,000 \qquad (4)$$

Cash Budget Constraints

The cash budget constraints recognize the amount of cash available each period, the cash outflow caused by the development of a particular use in that period, and also the cash inflows generated by a particular use in a given period. This set of constraints is rather complicated because the cash budget constraint for any period beyond the first period is dependent upon all of the previous constraints. In recognition of this difficulty, the constraints for the first two years are presented rather than a single constraint as was done above. The case example presented in this paper makes several simplifying assumptions. The budget is specified on a year-by-year basis rather than a month-to-month basis, which probably would be preferred in an actual application. Also, any profit center use which is developed in a given period is assumed to generate cash in that period. The example further assumes that \$2.5 million of equity capital is available in the first year and \$2.5 million is available in the second year. The cash inflows and outflows for each use in each period are obtained from the discounted cash flow analysis used to generate the contribution coefficients for the objective function. The first year budget constraint is specified as:

$$3.0564X_{11} - .5051X_{31} + 5.2497X_{51} + 10.7786X_{61} + .127X_{71} + .127X_{81} \leq 2,500,000 \quad (5)$$

In a similar manner the second and third year cash budget constraints are derived and presented in Equations 6 and 7, respectively.

$$2.642X_{11} + 3.0564X_{12} - .5051X_{31} - .5051X_{32} + 4.1401X_{51} + 5.2497X_{52} + 9.1912X_{61} + 10.7786X_{62} + .127X_{71} + .127X_{72} + .127X_{81} + .127X_{82} \leq 5,000,000 \quad (6)$$

$$2.2465X_{11} + 2.6420X_{12} + 3.0564X_{13} - .5051X_{31} - .5051X_{32} - .5051X_{33} - .7146X_{43} + 3.0984X_{51} + 4.1401X_{52} + 5.2497X_{53} + 7.712X_{61} + 9.1912X_{62} + 10.7786X_{63} + 1.127X_{71} + .127X_{72} + .127X_{73} + .127X_{81} + .127X_{82} + .127X_{93} + .127X_{103} \leq 5,000,000 \quad (7)$$

The Problem Set-Up

After the objective function criterion has been specified and the limiting conditions established, the overall problem can be stated. The problem statement for this example appears in Table 3 in the body of the paper.

At this point the problem must be restated so that it can be solved by the linear programming algorithm. The initial tableau is generated by adding the required slack and artificial variables to the constraints and then by including the necessary variables and appropriate contribution coefficients in the objective function.

FOOTNOTES

[1]The complete computations for all constraints and other abbreviated computations can be obtained by writing either author.

[2]The information used to specify the constraints in this subsection was obtained from Feasibility Study of the South Base Property, by Daniel Kohlhepp and Rodney Evans, loc. cit.

APPENDIX D

Recapitalization of Input Data
To Obtain the Present Value of Investor's Equity
Used in the Calculation of the Contribution Coefficients

Category	Garden Apartments	Townhouse	Single Family	Commercial Space	Office Space	Medical Space
1. Type property (0. = Residential, 1. = Commercial) =	0.0	0.0	0.0	1.0	1.0	1.0
2. Number of units in project =	625.	500.	100.	50000.	50000.	50000.
3. Average square footage per unit =	850.	1200.	1.0	1.0	1.0	1.0
4. Average monthly rental per unit =	220.	280.	4583.	.33	.5	.66
5. Expected occupancy =	.90	.95	1.0	.90	.90	.90
6. Annual growth rate of rental income over the holding period =	0.0	0.0	0.0	0.0	0.0	0.0
7. Total land cost =	82181.	84134.	.0001	13262.	14295.	14295.
8. Square foot cost of all improvements =	14.6	15.0	.0001	20.81	27.93	40.93
9. Required rate of return on equity[1] =	.08	.08	.08	.08	.08	.08
10. Operating cost as a percent of total rental income =	.40	.40	.76	.13	.38	.38
11. Annual growth rate of operating cost over holding period =	0.0	0.0	0.0	0.0	0.0	0.0
12. Depreciable life of improvements =	40.	40.	3.	20.	20.	20.
13. Depreciation method[2] =	2.0	2.0	1.0	1.5	1.5	1.5
14. Ordinary income tax rate =	.40	.40	.40	.40	.40	.40
15. Capital gains tax rate =	.20	.20	.20	.20	.20	.20
16. Holding period of the investment =	10.	10.	3.	10.	10.	10.
17. Annual growth rate of property value =	0.0	0.0	0.0	0.0	0.0	0.0
18. Selling commission (percent) =	.06	.06	0.0	.06	.06	.06
19. Investor's short-term borrowing rate =	.10	.10	.10	.10	.10	.10
20. Amount of loan 1 =	5982451.	6091228.	499500.	1039772.	1089244.	1437803.
21. Effective interest rate on loan 1 =	.10	.10	.10	.10	.10	.10
22. Amortization term of loan 1 =	25.	25.	3.0	25.	25.	25.
23. Does this project involve secondary financing =	NO	NO	NO	NO	NO	NO

[1] Originally a 12 percent required rate of return was used, but most of the land had negative net present values at that rate. Therefore, solely for the convenience of working with positive net present value figures, the authors reduced the required rate of return to 8 percent.

[2] 1. = Straight line; 1.25 = 125 percent; 1.5 = 150 percent; 2.0 = Double declining.